

Nonsingularity Under Data Rounding

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ABSTRACT

For a square floating point matrix A we give a test for the existence of a singular real matrix which rounds to A .

In this paper, which may be considered a footnote to [2], we show how a result concerning interval matrices can be applied to handle, at least theoretically, the problem of nonsingularity of real matrices in a floating point representation. We give the results for the decimal floating point system; however, the technique used applies easily to other systems (e.g. binary or hexadecimal) as well.

Let $A = (a_{ij})$ be an $n \times n$ floating point matrix in a decimal floating point system with normalized mantissa length d decimal digits. The matrix A represents in fact a whole class $C(A)$ of real matrices which round to A . Obviously, A can be considered numerically nonsingular if all matrices in $C(A)$ are nonsingular, and numerically singular if $C(A)$ contains a singular matrix, since in the latter case A cannot be distinguished from a singular matrix by means of our floating point system.

It follows from the properties of correct rounding that the class $C(A)$ can be described as

$$C(A) = \{A'; A - 10^{-d}A_2 \leq A' < A + 10^{-d}A_1\},$$

where, with the coefficients of A written in the form

$$a_{ij} = \pm m_{ij} 10^{p_{ij}} \quad (i, j = 1, \dots, n),$$

the matrices $A_1 = (a_{ij}^1)$, $A_2 = (a_{ij}^2)$ are given by

$$a_{ij}^1 = 0.5 \times 10^{p_{ij}} \quad (i, j = 1, \dots, n)$$

$$a_{ij}^2 = 0.5 \times 10^{p_{ij}} \quad \text{if } m_{ij} \in (0.1, 1)$$

$$a_{ij}^2 = 0.5 \times 10^{p_{ij}-1} \quad \text{if } m_{ij} = 0.1.$$

Let us denote

$$A_c = A + \frac{1}{2} 10^{-d} (A_1 - A_2),$$

$$A_r = \frac{1}{2} (A_1 + A_2),$$

so that $A_r \geq 0$. Then $C(A)$ can be written in a more appropriate centered form

$$C(A) = \{A'; A_c - 10^{-d} A_r \leq A' < A_c + 10^{-d} A_r\}.$$

Let us now introduce, assuming that A_c is nonsingular, the number

$$c(A) = \log_{10} \max\{\rho_0(A_c^{-1} S_1 A_r S_2); S_1, S_2 \text{ signature matrices}\}, \quad (1)$$

where a *signature matrix* is a diagonal matrix all of whose diagonal entries are equal to $+1$ or -1 , and for a square matrix B , $\rho_0(B)$ is defined by

$$\rho_0(B) = \max\{|\lambda|; Bx = \lambda x, x \neq 0, \lambda \text{ real}\}.$$

Hence ρ_0 is an analogue of the spectral radius of B , with the maximum being taken only over real eigenvalues; we define $\rho_0(B) = 0$ if no real eigenvalue exists.

We shall show in the next theorem that $c(A)$ is correctly defined and is closely connected to our problem of testing numerical (non)singularity:

THEOREM. *Let A be a floating point matrix such that A_c is nonsingular. Then $c(A)$ is well defined and we have:*

- (i) *if $c(A) < d$, then each matrix in $C(A)$ is nonsingular;*
- (ii) *if $c(A) > d$, then $C(A)$ contains a singular matrix.*

COMMENTS. The case of $c(A) = d$ remains undecided, but it can scarcely occur, since $c(A)$ is generally a real number, while d is integer. Also, notice that if no coefficient of A is of the form 10^p , p integer, then $A_c = A$, so that in this case the assumption of the theorem reduces to nonsingularity of A .

Proof. For a positive real number β , consider the interval matrix

$$M_\beta = \{A'; A_c - \beta A_r \leq A' \leq A_c + \beta A_r\}.$$

According to assertion (C3) of Theorem 5.1 in [2], M_β contains a singular matrix if and only if

$$\beta \rho_0(A_c^{-1} S_1 A_r S_2) \geq 1$$

holds for some signature matrices S_1, S_2 . Since $|A_c| \leq 2A_r$ (as can be easily verified), it follows that M_2 contains the zero matrix; thus $\rho_0(A_c^{-1} S_1 A_r S_2) \geq \frac{1}{2}$ for some signature matrices S_1, S_2 , which shows that the maximum in (1) is positive; hence $c(A)$ is *well defined*. Next we prove (i) and (ii)

(i) If $c(A) < d$, then $10^{-d} \rho_0(A_c^{-1} S_1 A_r S_2) < 1$ for any signature matrices S_1, S_2 , which in view of the above-quoted theorem shows that $M_{10^{-d}}$ consists only of nonsingular matrices. Since $C(A) \subset M_{10^{-d}}$, the assertion follows.

(ii) If $c(A) > d$, then $10^{-d} \rho_0(A_c^{-1} S_1 A_r S_2) > 1$ holds for some signature matrices S_1, S_2 . Take a $\beta \in (0, 10^{-d})$ such that $\beta \rho_0(A_c^{-1} S_1 A_r S_2) > 1$ still holds. Then M_β contains a singular matrix, and thus so does $C(A)$, since $M_\beta \subset C(A)$ in view of $\beta < 10^{-d}$. ■

The result, although solving the problem in principle, remains merely of theoretical interest due to the complicated form of the formula (1) (note that there are 2^n signature matrices of size n). Nevertheless, in the special case of nonnegative invertible matrices the formula for $c(A)$ can be given a very simple form (ρ denotes the spectral radius):

COROLLARY. Let $A_c^{-1} \geq 0$. Then we have

$$c(A) = \log_{10} \rho(A_c^{-1} A_r). \quad (2)$$

Proof. With E the unit matrix, from the Perron-Frobenius theorem

[1] we obtain that

$$\begin{aligned}
 c(A) &= \log_{10} \max_{S_1, S_2} \rho_0(A_c^{-1} S_1 A_r S_2) \\
 &\leq \log_{10} \max_{S_1, S_2} \rho(|A_c^{-1} S_1 A_r S_2|) \leq \log_{10} \rho(A_c^{-1} A_r) \\
 &= \log_{10} \rho_0(A_c^{-1} E A_r E) \leq c(A),
 \end{aligned}$$

which proves (2). ■

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REFERENCES

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- 2 J. Rohn, Systems of linear interval equations, *Linear Algebra Appl.* 26:39–78 (1989).

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